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The Equivalence of the Moduli of Continuity of the Best Approximation Operator and of Strong Unicity in L^1

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1. INTRODUCTION

Let X be a normed linear space and M_n a finite dimensional subspace of X. For $x \in X$ denote

$$\rho(x, M_n) = \inf\{\|x-q\| : q \in M_n\},$$

i.e., $\rho(x, M_n)$ is the distance from x to M_n. Denote P to be the possibly set-valued best approximation operator from X to M_n, namely,

$$P(x) = \{q \in M_n : \|x-q\| = \rho(x, M_n)\}$$

for each $x \in X$. Define the *local modulus of continuity* of P at $x \in X$ by

$$\Omega(M_n, x, \delta) = \sup\{\rho(q, P(x)) : q \in P(x_1),\ x_1 \in X,\ \|x-x_1\| \leqslant \delta\}.$$

(For a detailed discussion of this modulus, see [7].)

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Furthermore, assume that for a given $x \in X$, we have a sequence $\{q_k\} \subset M_n$ such that $\|x - q_k\| \to \rho(x, M_n)$ as $k \to \infty$. Then evidently $\rho(q_k, P(x)) \to 0$ as $k \to \infty$. This raises the natural question of how one estimates $\rho(q, P(x))$ when $\|x - q\| - \rho(x, M_n)$ is known. We are therefore led to the definition of the *local modulus of strong unicity* of P at $x \in X$:

$$\Omega^*(M_n, x, \delta) = \sup\{\rho(q, P(x)) \colon q \in M_n,\ \|x - q\| - \rho(x, M_n) \leqslant \delta\}.$$

Since M_n is finite dimensional, we have that $\Omega(M_n, x, \delta)$ and $\Omega^*(M_n, x, \delta)$ tend to zero as $\delta \to 0$.

The study of these moduli was inspired by the pioneering works of Freud [6], Newman and Shapiro [10], Holmes and Kripke [8], and Björnestål [4].

It can be easily seen that for any $M_n \subset X$ and $x \in X$

$$\Omega(M_n, x, \delta) \leqslant \Omega^*(M_n, x, 2\delta).$$

This follows from the inequality

$$\|x - q\| - \rho(x, M_n) \leqslant 2\delta,$$

which holds for any $q \in P(x_1)$ and $x_1 \in X$ such that $\|x - x_1\| \leqslant \delta$. Thus the modulus of continuity of P is at least of the same order as the modulus of strong unicity. In fact, it is known that in most spaces $\Omega(M_n, x, \delta)$ tends to zero faster than $\Omega^*(M_n, x, \delta)$ as $\delta \to 0$. This raises the question of whether these moduli can be of the same order for any $M_n \subset X$ and $x \in X$. This motivates the following.

Definition. We say that X satisfies the E-property if for any $M_n \subset X$ and $x \in X$ there exists a constant $\gamma > 0$ depending only on x and M_n such that

$$\Omega^*(M_n, x, \delta) \leqslant \Omega(M_n, x, \gamma\,\delta). \tag{1.1}$$

It is known that if $X = L^p$, then for any $M_n \subset L^p$ and $f \in L^p$, $\Omega(M_n f, \delta) \leqslant \text{const} \cdot \delta$ if $2 \leqslant p < \infty$ [8] and $\Omega(M_n, f, \delta \leqslant \text{const.} \cdot \delta^{p/2}$, $0 < \delta \leqslant 1$, if $1 < p < 2$ [4]. On the other hand, $\Omega^*(M_n, f, \delta)$ cannot, in general, be of better order than $\delta^{1/2}$, $1 < p < \infty$ [1]. Thus we can conclude that L^p, $1 < p < \infty$, does not satisfy the E-property.

In the present paper we shall prove tha L^1 satisfies the E-property. We remark here that in [9] it was shown that if $X = L^1[a, b]$ with Lebesque measure and M_n is a Haar subspace of $C[a, b]$, then (1.1) holds.

2. The Equivalence in L^1

Let (T, Σ, μ) be a complete, σ-finite, positive measure space. Consider $X = L^1 = L^1(T, \Sigma, \mu)$ be the Banach space of all equivalence classes of real-valued, μ-integrable functions normed in the usual way.

THEOREM 1. *L^1 satisfies the E-property.*

Proof. Let $M_n \subset L^1$ and $f \in L^1 \backslash M_n$ be arbitrary. (The case where $f \in M_n$ is trivial.) Without loss of generality, we may assume that $0 \in P(f)$. By a characterization theorem proved in [13], $0 \in P(f)$ if and only if there exists a function $\phi \in L^\infty$, $|\phi| \leqslant 1$, such that

$$\int_{\operatorname{supp}(f)} p \operatorname{sign}(f)\, d\mu + \int_{Z(f)} p\phi\, d\mu = 0 \qquad (p \in M_n), \tag{2.1}$$

where $Z(f) = \{t \in T\colon f(t) = 0\}$ and $\operatorname{supp}(f) = T \backslash Z(f)$. Here we have used the assumption that (T, Σ, μ) is σ-finite. However, no generality is lost since T can be replaced with the cumulative support of f and M_n.

Consider a $q \in M_n$ such that

$$\|f - q\| - \rho(f, M_n) \leqslant \delta. \tag{2.2}$$

Let $Z_1(f) = \{t \in Z(f)\colon |\phi(t)| = 1\}$, $Z_2(f) = Z(f) \backslash Z_1(f)$, and set

$$A(q) = \{t \in T\colon 0 < f(t) \leqslant q(t) \text{ or } q(t) \leqslant f(t) < 0\}$$

and

$$B(q) = T \backslash (A(q) \cup Z(f)).$$

Consider now the function $f_1 \in L^1$ defined by

$$\begin{aligned} f_1(t) &= q(t), && t \in A(q) \cup Z_2(f), \\ &= |q(t)|\phi(t) + q(t), && t \in Z_1(f), \\ &= f(t), && t \in B(q). \end{aligned} \tag{2.3}$$

Then $Z(f_1 - q) = A(q) \cup Z_2(f) \cup (Z_1(f) \cap Z(q))$ and since $\operatorname{sign}(f) = \operatorname{sign}(f - q)$ on $B(q)$, we have

$$\begin{aligned} \operatorname{sign}(f_1 - q)(t) &= 0, && t \in Z(f_1 - q), \\ &= \phi(t), && t \in Z_1(f) \backslash Z(q), \\ &= \operatorname{sign}(f(t)), && t \in B(q). \end{aligned} \tag{2.4}$$

Set

$$\begin{aligned}\phi_1(t) &= \operatorname{sign}(f(t)), && t \in A(q),\\ &= \phi(t), && t \in Z_2(f) \cup (Z_1(f) \cap Z(q)),\\ &= 1, && t \in T \setminus Z(f_1 - q).\end{aligned}$$

Then $\phi_1 \in L^\infty$, $|\phi_1| \leqslant 1$, and by (2.1) and (2.4), we have for any $p \in M_n$,

$$\begin{aligned}&\int_{\operatorname{supp}(f_1 - q)} p \operatorname{sign}(f_1 - q)\, d\mu + \int_{Z(f_1 - q)} p\phi_1\, d\mu\\ &\quad = \int_{Z_1(f) \setminus Z(q)} p\phi\, d\mu + \int_{B(q)} p \operatorname{sign}(f)\, d\mu\\ &\qquad + \int_{A(q)} p \operatorname{sign}(f)\, d\mu + \int_{Z_2(f) \cup (Z_1(f) \cap Z(q))} p\phi\, d\mu\\ &\quad = \int_{\operatorname{supp}(f)} p \operatorname{sign}(f)\, d\mu + \int_{Z(f)} p\phi\, d\mu = 0.\end{aligned}$$

Therefore $0 \in P(f_1 - q)$, i.e., $q \in P(f_1)$.

We now give an estimate for $\|f - f_1\|_1$. By (2.1) and (2.2)

$$\begin{aligned}\delta &\geqslant \|f - q\|_1 - \rho(f, M_n) = \|f - q\|_1 - \|f\|_1\\ &= \int_T |f - q|\, d\mu - \int_{\operatorname{supp}(f)} |f|\, d\mu\\ &= \int_{\operatorname{supp}(f)} |f - q|\, d\mu - \int_{\operatorname{supp}(f),} (f - q) \operatorname{sign}(f)\, d\mu + \int_{Z(f)} \{|q| + q\phi\}\, d\mu\\ &= 2\int_{A(q)} |f - q|\, d\mu + \int_{Z(f)} \{|q| + q\phi\}\, d\mu. \qquad (2.5)\end{aligned}$$

On the other hand, by the construction of f_1, we have

$$\begin{aligned}\|f - f_1\|_1 &= \int_T |f - f_1|\, d\mu\\ &= \int_{A(q)} |f - q|\, d\mu + \int_{Z_2(f)} |q|\, d\mu + \int_{Z_1(f)} \big||q|\phi + q\big|\, d\mu\\ &= \int_{A(q)} |f - q|\, d\mu + \int_{Z_1(f)} \{|q| + \phi q\}\, d\mu + \int_{Z_2(f)} |q|\, d\mu. \qquad (2.6)\end{aligned}$$

Set $M_k=\{p\in M_n\colon p=0\ \mu\text{-a.e. on } Z_2(f)\}$. Now M_k is a subspace of M_n of some dimension k, $0\leqslant k\leqslant n$. Then $M_n=\operatorname{span}\{p_1,\dots,p_k,p_{k+1},\dots,p_n\}$, where $\{p_1,\dots,p_k\}$ is a basis for M_k. Set $M_{n-k}=\operatorname{span}\{p_{k+1},\dots,p_n\}$. Evidently,

$$\|p\|_*=\int_{Z_2(f)}|p|\,d\mu \quad\text{and}\quad \|p\|_{**}=\int_{Z_2(f)}(1-|\phi|)|p|\,d\mu$$

are two different norms on M_{n-k}. Thus, by the equivalence of norms on finite dimensional spaces, there exists a $\gamma\geqslant 1$ depending only on f and M_n such that for any $p\in M_{n-k}$

$$\gamma\int_{Z_2(f)}(1-|\phi|)|p|\,d\mu\geqslant\int_{Z_2(f)}|p|\,d\mu. \tag{2.7}$$

Then relation (2.7) also holds for each $p\in M_n$. This, (2.6), and (2.5) imply that

$$\begin{aligned}
\|f-f_1\|_1&\leqslant\int_{A(q)}|f-q|\,d\mu+\int_{Z_1(f)}\{|q|+\phi q\}\,d\mu+\gamma\int_{Z_2(f)}(1-|\phi|)|q|\,d\mu\\
&\leqslant\int_{A(q)}|f-q|\,d\mu+\int_{Z_1(f)}\{|q|+\phi q\}\,d\mu+\gamma\int_{Z_2(f)}\{|q|+\phi q\}\,d\mu\\
&\leqslant\gamma\left\{2\int_{A(q)}|f-q|\,d\mu+\int_{Z(f)}\{|q|+\phi q\}\,d\mu\right\}\\
&\leqslant\gamma\delta.
\end{aligned}$$

Finally, since $q\in P(f_1)$ it follows that

$$\rho(q,P(f))\leqslant\Omega(M_n,f,\gamma\delta),$$

i.e.,

$$\Omega^*(M_n,f,\delta)\leqslant\Omega(M_n,f,\gamma\delta) \tag{2.8}$$

and the theorem is proved. ∎

Remark 1. The proof of the above theorem provides an explicit form of the constant γ in (2.8). Let $M_n\subset L^1$, $f\in L^1\backslash M_n$ and choose $p\in P(f)$ and any $\phi\in L^\infty$ such that (2.2) holds for ϕ, $f-p$, and any $q\in M_n$. Then

(i) if $\mu(Z_2(f-p))>0$, then

$$\gamma=\sup_{q\in M_{n-k}\backslash\{0\}}\frac{\int_{Z_2(f-p)}|q|\,d\mu}{\int_{Z_2(f-p)}(1-|\phi|)|q|\,d\mu},$$

where M_{n-k} is a subspace complement to $M_k=\{q\in M_n\colon q=0\ \mu\text{-a.e. on } Z_2(f-p)\}$; or

(ii) if $\mu(Z_2(f-p))=0$, then $\gamma=1$.

We now show that in the event (T, Σ, μ) is nonatomic the constant γ can always be taken to be one. In order to prove this, we will require the following lemma which can be found in [12].

LEMMA. *Suppose that the measure space (T, Σ, μ) is nonatomic, that ϕ is a measurable function on T with $|\phi|\leqslant 1$ μ–a.e., and that $q_1, q_2,..., q_n$ are in L^1. Then there exists a measurable function ψ on T with $|\psi|=1$ μ–a.e., such that*

$$\int_T \psi q_i\, d\mu=\int_T \phi q_i\, d\mu,\ i=1, 2,..., n.$$

We then have the following:

THEOREM 2. *Suppose that the measure space (T, Σ, μ) is nonatomic. Then L^1 satisfies the E-property with $\gamma=1$.*

Proof. We need only note that for $f\in L^1\backslash M_n$ and $p\in P(f)$, the above lemma implies we can choose a $\phi\in L^\infty$ with $|\phi|=1$ such that (2.2) holds for ϕ, $f-p$, and all $q\in M_n$. This implies that $\mu(Z_2(f-p))=0$ and so we can take $\gamma=1$ in (2.8). ▮

We now give some additional definitions. We say that the best approximation operator P satisfies a *Lipschitz condition* at $f\in L^1$ if f has a unique best approximation $p^*(f)\in M_n$ and there exists a constant $\lambda>0$ depending only on f and M_n such that for every $f_1\in L^1$,

$$\sup\{\|p^*(f)-q\|_1 : q\in P(f_1)\}\leqslant \lambda\|f-f_1\|_1. \tag{2.9}$$

Furthermore, we say that $p^*(f)$ is a *strongly unique* best approximation if for some $\gamma>0$ depending only on f and M_n and any $q\in M_n$,

$$\|q-p^*(f)\|_1\leqslant \gamma\{\|f-q\|_1-\|f-p^*(f)\|_1\}. \tag{2.10}$$

Evidently (see [5, p. 82], e.g.), the strong unicity property implies the Lipschitz property. It is known [11] that $p^*(f)$ is the strongly unique best approximation to $f\in L^1$ if and only if

$$\left|\int_{\operatorname{supp}(f-p^*(f))} q\operatorname{sign}(f-p^*(f))\, d\mu\right|<\int_{Z(f-p^*(f))}|q|\, d\mu$$

for each $q\in M_n$.

By the results proved in [1, 4, and 8], the Lipschitz property and strong unicity are in general not equivalent in L^p, $1 < p < \infty$. Namely, the class of functions which have P satisfying the Lipschitz property may be, in these spaces, a much wider class than that which have strongly unique best approximations.

However, in view of the above theorems, we obtain that strong unicity and the Lipschitz property are equivalent in L^1.

COROLLARY 1. *For any $M_n \subset L^1$ and $f \in L^1$, the following are equivalent:*

(i) *f has a strongly unique best approximation from M_n;*

(ii) *the best approximation operator satisfies the Lipschitz property at f;*

(iii) *if $p^*(f)$ is the best approximation of f, then for any $q \in M_n$*

$$\left| \int_{\operatorname{supp}(f - p^*(f))} q \operatorname{sign}(f - p^*(f))\, d\mu \right| < \int_{Z(f - p^*(f))} |q|\, d\mu.$$

Remark 2. Corollary 1, under the assumption that (T, Σ, μ) be nonatomic, coupled with a result in [2], implies that the set of functions where P satisfies the Lipschitz property is dense.

Remark 3. In the space C, the continuous real-valued functions on a compact metric space, Bartelt and Schmidt [3] proved that the Lipschitz property and strong unicity are equivalent. However, it is not known whether C satisfies the E-property.

Finally, we make note of the following. If P satisfies the Lipschitz property at $f \in L^1$, then we define the Lipschitz constant, $\lambda_n(f)$, to be the largest constant such that (2.9) holds for all $f_1 \in L^1$. If $f \in L^1$ has a strongly unique best approximation from M_n, then we define the strong unicity constant, $\gamma_n(f)$, to be the largest constant such that (2.10) holds for all $q \in M_n$. Evidently, since $\Omega(M_n, f, \delta) \leqslant \Omega^*(M_n, f, 2\delta)$, we have $\lambda_n(f) \leqslant 2\gamma_n(f)$, but in light of Theorem 2, we have the following corollary.

COROLLARY 2. *Suppose that the measure space (T, Σ, μ) is nonatomic. Let $M_n \subset L^1$ and $f \in L^1 \backslash M_n$. If f has a strongly unique best approximation from M_n, then*

$$\gamma_n(f) \leqslant \lambda_n(f) \leqslant 2\gamma_n(f). \tag{2.11}$$

Remark 4. In the space C, it is known that (2.11) cannot in general hold since $\gamma_n(f)$ can tend to infinity faster than $\lambda_n(f)$.

3. CONCLUSIONS

In the present paper, we have shown that the modulus of continuity of P and the modulus of strong unicity are of the same order for every finite-dimensional subspace of L^1. This result was then used to show that the Lipschitz property of P is equivalent to strong unicity in L^1. This, with the results in [3] for C, show that these two spaces are very special from the point of view of approximation theory. A question comes to mind as to whether there are other spaces where the Lipschitz property of P and strong unicity are equivalent for every finite dimensional subspace. Furthermore, do such spaces necessarily have to be nonstrictly convex?

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